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On Caputo Delta $q-$Fractional Dynamical Systems: Lyapunov Stability

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Abstract

The investigation of dynamic systems that incorporate Caputo delta $q-$fractional derivatives has garnered significant interest due to their practicality in diverse scientific and engineering fields. This paper studies the stability of a dynamic system with the Caputo delta $q-$fractional derivative using Lyapunov's direct method. The motivation behind our work stems from the necessity to comprehend the dynamics and resilience of systems defined by Caputo delta $q-$fractional derivatives, which exemplify a category of operators that are both non-local and non-singular. This unique fractional derivative, which accounts for memory effects and long-range interactions, adds a level of complexity that calls for a thorough study of stability properties. Expanding upon previous scholarly works, we fill a significant research void by presenting a series of criteria that determine the stability, asymptotic stability, and uniform stability of dynamic systems with Caputo delta $q-$fractional derivatives. Through the utilization of Lyapunov's direct method, we establish a meticulous framework for examining the stability of these systems, providing a valuable understanding of their dynamic behavior.

Keywords: stability analysis; Lyapunov's direct method; asymptotic stability; uniform stability; delta-fractional derivatives; $q-$calculus; $q-$fractional; time scale calculus.

1 Introduction

The field of fractional calculus encompasses the study of derivatives and integrals of non-integer order [8]. This particular region has attracted considerable interest owing to its notable significance in multiple scientific and engineering fields [9]. The concept of $q-$calculus was introduced by Jackson in the early 20th century. This field of study focuses on the exploration of calculus principles without the use of limits [11]. The pioneering investigation of $q-$fractional integrals and derivatives was carried out by Al-Salam [3] and Agarwal [2] . The area of $q-$fractional calculus has garnered considerable interest owing to its capacity to establish a connection between the principles of fractional calculus [13] and $q-$calculus [4].

In recent years, scholars have increasingly focused on the temporal dimension in mathematical modelling, emphasising the application of fractional calculus and $q-$fractional calculus methods. These approaches have proven valuable in modelling various biological and physical processes, including the spread of infectious diseases [7], in-host tuberculosis dynamics [8], Ebola virus [1], and many other modelling. Such advancements highlight the versatility of fractional and $q-$fractional methods in addressing complex, real-world systems with memory effects and anomalous diffusion behaviours [16]. The motivation for this specific interest stems from the study of time scale calculus, as demonstrated by the scholarly inquiries conducted by [5, 12]. In addition, numerous scholars have undertaken research on the integration of time scale and $q-$fractional calculus approaches, drawing inspiration from the contributions of Al-Salam [3], Agarwal [2] and other scholars [14]. The results hold importance in the investigation of fractional calculus in the time scale represented as $\mathbb{T}_q := \{q^{\varpi} : \varpi \in \mathbb{Z}\} \cup \{0\}$, where $0 < q < 1$. Considerable scholarly inquiry has been devoted to the analysis and investigation of the Caputo nabla operator. The examination of the fractional dynamical equation has been conducted by many researchers [15].

There are many advantages of the fractional order derivatives over the classical ones in the description of many real-world dynamical systems [9]. It was found that a dynamical system with a classical derivative may not be stable, but that the same system may be stable if the classical derivative is replaced with a fractional derivative. In other words, the region of stability in the fractional dynamical system is bigger than the region of the corresponding dynamical system. It turns out that the identical logic still applies to $q-$fractional difference systems.

The Lyapunov stability criteria are a main method to analysis the stability of nonlinear dynamical systems without solving them. Unfortunately, Lyapunov stability criteria cannot be applied for fractional dynamic systems since they require applying the Leibniz rule, which does not satisfy for the Caputo fractional derivative. This reason has motivated many scholars to introduce a few algebraic criteria for studying these types of dynamic systems [10]. However, there are some attempts and results to investigate the stability of fractional dynamic systems by the Lyapunov direct method [6] and the references therein. According to our good knowledge, there is no study about the stability of dynamic system with Caputo delta $q-$fractional derivative. Therefore, this paper will address this important issue.

2 Preliminaries

This section covers some fundamental $q-$time scale calculus concepts.

Let us consider the time scale $\mathbb{T}_q$, where $0 < q < 1$.

$$\mathbb{T}_q = \{q^{\varpi} : \varpi \in \mathbb{Z}\} \cup \{0\},$$

in which $\mathbb{Z}$ denotes the set of integers.

This paper presents an explanation of the delta $q-$derivative concept for the function $g : \mathbb{T}_q \to \mathbb{R}$.

$$\Delta_q g(\varpi) = \frac{g(q\varpi) - g(\varpi)}{(q-1)\varpi}, \qquad \varpi \in \mathbb{T}_q \backslash \{0\}. \tag{1}$$

The higher-order delta $q-$derivatives may be defined in the following manner:

$$\Delta_q^0 g(\varpi) = g(\varpi), \qquad \Delta_q^{\ell} g(\varpi) = \Delta_q(\Delta_q^{\ell-1} g(\varpi)), \quad (\ell = 1, 2, 3, \ldots). \tag{2}$$

The expression for the delta $q-$integral of the function $g(\varpi)$ is provided as follows:

$$(I_{q,0}g)(\varpi) = \int_0^{\varpi} g(\omega)\Delta_q\omega = (1-q)\sum_{\ell=0}^{\infty} \varpi q^{\ell} g(\varpi q^{\ell}), \qquad \varpi \in \mathbb{T}_{q,}, \tag{3}$$

and

$$(I_{q,a}g)(\varpi) = \int_a^{\varpi} g(\omega)\Delta_q\omega = \int_0^{\varpi} g(\omega)\Delta_q\omega - \int_0^{a} g(\omega)\Delta_q\omega, \qquad a, \varpi \in \mathbb{T}_q. \tag{4}$$

The essential delta $q-$calculus theorem presents,

$$\Delta_q \int_0^{\varpi} g(\omega)\Delta_q\omega = g(\varpi). \tag{5}$$

Additionally, in the case where the function $g(\varpi)$ exhibits continuity at zero,

$$\int_0^{\varpi} \Delta_q g(\omega)\Delta_q\omega = g(\varpi) - g(0). \tag{6}$$

The subsequent identities will also prove to be advantageous.

$$\Delta_q \int_a^{\varpi} g(\varpi,\omega)\Delta_q\omega = \int_a^{\varpi} \Delta_q g(\varpi,\omega)\Delta_q\omega + g(q\varpi,\varpi), \tag{7}$$

and

$$\Delta_q \int_{\varpi}^{b} g(\varpi,\omega)\Delta_q\omega = \int_{q\varpi}^{b} \Delta_q g(\varpi,\omega)\Delta_q\omega - g(\varpi,\varpi). \tag{8}$$

Definition 2.1. *The delta $q-$factorial function for $\vartheta \in \mathbb{N}$ is defined as,*

$$(\varpi - \omega)_q^{(0)} = 1, \qquad (\varpi - \omega)_q^{(\vartheta)} = \prod_{r=1}^{\vartheta-1} (\varpi - q^r\omega). \tag{9}$$

Also,

$$(r - \omega)_q^{(\alpha)} = \varpi^{\alpha} \prod_{r=0}^{\infty} \frac{1 - \frac{\omega}{\varpi} q^r}{1 - \frac{\omega}{\varpi} q^{r+\alpha}}, \tag{10}$$

where α be a positive real number that is also an integer.

Definition 2.2. *The definition of the delta $q-$Gamma function is*

$$\Gamma_q(\gamma) = (1-q)_q^{(\gamma-1)}(1-q)^{1-\gamma}, \qquad \forall\, \gamma \in \mathbb{C}\backslash \{-\eta, \eta \in \mathbb{N}_0\}, \tag{11}$$

that fulfills,

$$\Gamma_q(1+\gamma) = \frac{1-q^\gamma}{1-q}\Gamma_q(\gamma), \qquad \Gamma_q(1) = 1. \tag{12}$$

Definition 2.3. *The space $[e, f]$ is defined as the set of all continuous functions that possess continuous delta $q-$derivatives up to order $\ell - 1$,*

$$\mathcal{C}_q^{(\ell)}[0, f] = \left\{g(\varpi) : \Delta_q^\eta g(\varpi) \in \mathcal{C}[0, f], \qquad \forall\, \eta = 0, 1, \ldots, \ell\right\}. \tag{13}$$

Definition 2.4. *Let $\alpha > 0$, $\varpi, \varpi_0 \in \mathbb{T}_q$. The fractional delta $q-$integral for the function $g : \mathbb{T}_q \to \mathbb{R}$ is*

$$\begin{aligned} &I^0_{\Delta_q,\varpi_0} g(\varpi) = g(\varpi), \\ &I^\alpha_{\Delta_q,\varpi_0} g(\varpi) = \frac{1}{\Gamma_q(\alpha)} \int_{\varpi_0}^{\varpi} (\varpi - q\omega)_q^{(\alpha-1)} g(\omega) \Delta_q \omega. \end{aligned} \tag{14}$$

For $\alpha_1, \alpha_2 > 0$, then,

$$\left(I^{\alpha_2}_{\Delta_q,\varpi_0} I^{\alpha_1}_{\Delta_q,\varpi_0} g\right)(\varpi) = \left(I^{\alpha_1+\alpha_2}_{\Delta_q,\varpi_0} g\right)(\varpi), \qquad \varpi_0 < \varpi. \tag{15}$$

Lemma 2.1. *Assuming $\gamma, \delta \in \mathbb{R}$, we obtain*

1. $(\varpi - \omega)_q^{(\gamma+\delta)} = (\varpi - \omega)_q^{(\gamma)} (\varpi - q^\gamma \omega)_q^{(\delta)}$.
2. $(b\varpi - b\omega)_q^{(\gamma)} = b^\gamma (\varpi - \omega)_q^{(\gamma)}$.
3. $\Delta_q (\varpi - \omega)_q^{(\beta)} = \dfrac{1-q^\gamma}{1-q} (\varpi - \omega)_q^{(\gamma-1)}$.
4. $\Delta_q (\varpi - \omega)_q^{(\gamma)} = -\dfrac{1-q^\beta}{1-q} (\varpi - q\omega)_q^{(\gamma-1)}$.

Definition 2.5. *Let $\varpi, \varpi_0 \in \mathbb{T}_q$. The formula representing fractional delta $q-$derivative in the sense of Riemann-Lioville for the function $g : \mathbb{T}_q \to \mathbb{R}$ can be expressed as follows:*

$$D^\alpha_{\Delta_q,\varpi_0} g(\varpi) = \Delta_q^\eta I^{\eta-\alpha}_{\Delta_q,\varpi_0} g(\varpi), \tag{16}$$

such that $\alpha \geq 0$ and $\eta = [\alpha] + 1$.

For $\alpha \in \mathbb{R}^+$, $0 < \varpi_0 < \varpi$. Then,

$$D^\alpha_{\Delta_q,\varpi_0} I^\alpha_{\Delta_q,\varpi_0} g(\varpi) = g(\varpi). \tag{17}$$

Definition 2.6. *Let $\varpi, \varpi_0 \in \mathbb{T}_q$. The formula representing fractional delta $q-$derivative in the sense of Caputo for the function $g : \mathbb{T}_q \to \mathbb{R}$ can be expressed as follows:*

$$\begin{aligned} {}^C D^\alpha_{\Delta_q,\varpi_0} g(\varpi) &= I^{\eta-\alpha}_{\Delta_q,\varpi_0} \Delta_q^\eta g(\varpi), \\ &= \frac{1}{\Gamma_q(\eta-\alpha)} \int_{\varpi_0}^{\varpi} (\varpi - q\omega)_q^{(\eta-\alpha-1)} \Delta_q^\eta g(\omega) \Delta_q \omega, \end{aligned} \tag{18}$$

where $\eta = [\alpha] + 1$ and $\alpha \geq 0$.

Theorem 2.1. *Let $0 < \alpha < 1$, we obtain*

$$^{C}D^{\alpha}_{\Delta_q,\varpi_0} g(\varpi) = D^{\alpha}_{\Delta_q,\varpi_0} g(\varpi) - \frac{(\varpi - \varpi_0)_q^{-\alpha}}{\Gamma_q(1-\alpha)} g(\varpi_0).$$

Lemma 2.2. *Let $\alpha > 0$, and $g : \mathbb{T}_q \to \mathbb{R}$ is defined in appropriate domains. Then,*

$$I^{\alpha}_{\Delta_q,\varpi_0} {}^{C}D^{\alpha}_{\Delta_q,\varpi_0} g(\varpi) = g(\varpi) - \sum_{\ell=0}^{\eta-1} \frac{(\varpi - \varpi_0)_q^{(\ell)}}{\Gamma_q(\ell+1)} \Delta_q^{\ell} g(\varpi_0), \tag{19}$$

and if $0 < \alpha \leq 1$, then,

$$I^{\alpha}_{\Delta_q,\varpi_0} {}^{C}D^{\alpha}_{\Delta_q,\varpi_0} g(\varpi) = g(\varpi) - g(\varpi_0). \tag{20}$$

Furthermore, we will employ the subsequent identity:

$$I^{\alpha}_{\Delta_q,\varpi_0} (x - \varpi_0)_q^{\upsilon} = \frac{\Gamma_q(\upsilon+1)}{\Gamma_q(\alpha+\upsilon+1)} (x - \varpi_0)_q^{\upsilon+\alpha}, \qquad 0 < \varpi_0 < x < \varpi, \tag{21}$$

where $\alpha \in \mathbb{R}^{+}$ and $\upsilon \in (-1, \infty)$.

3 Main Results

On time scale, the direct method of Lyapunov will adopted in this section to investigate the stability of dynamic system with Caputo delta $q-$fractional derivative.

$$\begin{aligned} ^{C}D^{\alpha}_{\Delta_q,\varpi_0} \xi(\varpi) &= g(\varpi, \xi(\varpi)), \\ \xi(\varpi_0) &= \xi_0, \end{aligned} \tag{22}$$

where $g : \mathbb{T}_q \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $\alpha \in (0, 1)$, $\varpi \geq \varpi_0$, and $\varpi_0 \in \mathbb{T}_q$.

For all $\varpi \in \mathbb{T}_q$, let $g(\varpi, 0) = 0$, as a result, the system (22) approves the trivial solution. In studying the system's properties (22), the following are listed:

Definition 3.1. *The stationary point of (22), $\xi(\varpi) = 0$, is defined as,*

1. *Stable if $\gamma = \gamma(\varepsilon, \varpi_0) > 0$ exists for each $\varepsilon > 0$ and $\varpi_0 \in \mathbb{T}_q$ such that if $\|\xi_0\| < \gamma$ we have $\|\xi(\varpi)\| < \varepsilon$, for all $\varpi \in \mathbb{T}_q$, $\varpi \geq \varpi_0$.*
2. *Uniformly stable if it is stable and γ depends only on ε.*
3. *asymptotically stable, if it is stable and there exists $\gamma = \gamma(\varpi_0) > 0$, such that if $\|\xi_0\| < \gamma$ implies that $\lim\limits_{\varpi \to \infty} \xi(\varpi, \varpi_0, \xi_0) = 0$.*

Definition 3.2. *The function $\Xi(\ell)$ is referred to as being of class $\mathcal{K}$ iff $\Xi \in \mathcal{C}\left[[0, \eta), \mathbb{R}_+\right]$, where $\eta \in \mathbb{R}_+$, $\Xi(0) = 0$ and $\Xi(\ell)$ increases strictly monotonically in ℓ.*

Definition 3.3. *The scalar function $V(\varpi,\xi) : \mathbb{T}_q \times \mathcal{S}_\eta \to \mathbb{R}$, where $\mathcal{S}_\eta = \{\xi \in \mathbb{R}^n : \|\xi\| < \eta\}$, is called to be positive definite iff,*

$$V(\varpi, 0) = 0, \qquad \forall\, \varpi \in \mathbb{T}_q,$$

and $\Xi(\ell) \in \mathcal{K}$ such that,

$$\Xi(\ell) \leq V(\varpi, \xi), \|\xi\| = \ell, \qquad \forall\ (\varpi, \xi) \in \mathbb{T}_q \times S_\eta.$$

Definition 3.4. *A scalar function $V(\varpi,\xi) : \mathbb{T}_q \times \mathcal{S}_\eta \to \mathbb{R}$, where $\mathcal{S}_\eta = \{\xi \in \mathbb{R}^n : \|\xi\| < \eta\}$, is called decreasing function iff,*

$$V(\varpi, 0) = 0, \qquad \forall\, \varpi \in \mathbb{T}_q,$$

and $\Xi(\ell) \in \mathcal{K}$ such that,

$$V(\varpi,\xi) \leq \Xi(\ell),\ \|\xi\| = \ell, \qquad \forall\, (\varpi,\xi) \in \mathbb{T}_q \times \mathcal{S}_\eta.$$

Now, the stability of system (22) will be discussed in the following theorems,

Theorem 3.1. *The stationary point of (22) is stable, if there is a positive definite scalar function, $V(\varpi,\xi) \in \mathcal{C}[\mathbb{T}_q \times \mathcal{S}_\eta, \mathbb{R}_+]$, where*

$$^{C}D^{\alpha}_{\Delta_q,\varpi_0} V(\varpi,\xi(\varpi)) \leq 0, \qquad \forall\, (\varpi,\xi) \in \mathbb{T}_q \times \mathcal{S}_\eta.$$

Proof. Let $\xi(\varpi) = \xi(\varpi,\varpi_0,\xi_0)$ be the system's solution (22). Due to $V(\varpi,\xi)$ being a positive definite, there exists a function $\Xi \in \mathcal{K}$ such that,

$$\Xi(\|\xi\|) \leq V(\varpi,\xi), \qquad \forall\, (\varpi,\xi) \in \mathbb{T}_q \times \mathcal{S}_\eta.$$

For any $0 < \varepsilon < \eta$, $\varepsilon > 0$, one may choose a $\gamma = \gamma(\varpi_0, \varepsilon)$ such that,

$$\|\xi\| < \gamma \Rightarrow V(\varpi_0,\xi_0) < \Xi(\varepsilon).$$

This choice is possible since $V(\varpi_0, 0) = 0$ and $V(t_0,\xi)$ is continuous in ξ.

For each solution to (22), since $^{C}D^{\alpha}_{\Delta_q,\varpi_0} V(\varpi,\xi(\varpi)) \leq 0$, and (20), we have

$$V(\varpi,\xi(\varpi)) \leq V(\varpi_0,\xi_0), \qquad \forall\, \varpi \geq \varpi_0.$$

As a result, we get

$$\Xi(\|\xi(\varpi)\|) \leq V(\varpi,\xi(\varpi)) \leq V(\varpi_0,\xi_0) < \Xi(\varepsilon).$$

Since $\Xi \in \mathcal{K}$, we obtain

$$\|\xi(\varpi)\| < \varepsilon, \qquad \forall\, \varpi \geq \varpi_0.$$

□

Theorem 3.2. *The stationary point of (22) is uniformly stable, if a decreasing function and positive definite $V(\varpi,\xi) \in \mathcal{C}[\mathbb{T}_q \times S_\eta, \mathbb{R}_+]$ exists such that,*

$$^{C}D^{\alpha}_{\Delta_q,\varpi_0} V(\varpi,\xi(\varpi)) \leq 0, \qquad \forall(\varpi,\xi) \in \mathbb{T}_q \times S_\eta.$$

Proof. Let $\xi(\varpi) = \xi(\varpi,\varpi_0,\xi_0)$ be a solution of system (22). Since $V(\varpi,\xi)$ is decreasing function and positive definite, there exists Ξ, $\Psi \in \mathcal{K}$ such that,

$$\Xi(\|\xi\|) \leq V(\varpi,\xi) \leq \Psi(\|\xi\|), \qquad \forall(\varpi,\xi) \in \mathbb{T}_q \times S_\eta.$$

For each $0 < \varepsilon < \eta$, $\varepsilon > 0$, one may choose a $\gamma = \gamma(\varepsilon)$ such that $\Psi(\gamma) < \Xi(\varepsilon)$. Since Ψ is continuous and $\Psi(0) = 0$, this choice is possible.

For any solution to (22), we get

$$\Xi(\|\xi(\varpi)\|) \le V(\varpi, \xi(\varpi)),$$

with $\|\xi_0\| < \gamma(\varepsilon)$.

Using (20), and since ${}^{C}D^{\alpha}_{\Delta_q,\varpi_0} V(\varpi, \xi(\varpi)) \le 0$, we get

$$V(\varpi, \xi(\varpi)) \le V(\varpi_0, \xi_0), \qquad \forall\, \varpi \in \mathbb{T}_q.$$

Consequently, we have

$$\Xi(\|\xi(\varpi)\|) \le V(\varpi, \xi(\varpi)) \le V(\varpi_0, \xi_0) \le \Xi(\|\xi_0\|) < \Psi(\gamma) < \Xi(\varepsilon),$$

and thus $\|\xi(\varpi)\| < \varepsilon$, at all $t \ge \varpi_0$, $t \in \mathbb{T}_q$. □

Theorem 3.3. *The stationary point of (22) is asymptotically stable, if there is a positive definite function $V(\varpi, \xi) \in \mathcal{C}[\mathbb{T}_q \times \mathcal{S}_\eta, \mathbb{R}_+]$ where*

$${}^{C}D^{\alpha}_{\Delta_q,\varpi_0} V(\varpi, \xi(\varpi)) \le -\varphi(V(\varpi, \xi(\varpi))), \qquad \forall\, (\varpi, \xi) \in \mathbb{T}_q \times S_\eta, \qquad \varpi \ge \varpi_0,$$

for all $\varphi \in \mathcal{K}$.

Proof. Clearly, Theorem 3.2 are fulfilled, and the stationary point of (22) is stable.

Since

$${}^{C}D^{\alpha}_{\Delta_q,\varpi_0} V(\varpi, \xi(\varpi)) \le -\varphi(V(\varpi, \xi)), \qquad \forall\, (\varpi, \xi) \in \mathbb{T}_q \times S_\eta.$$

Using (20), we get

$$V(\varpi, \xi) \le (V(\varpi_0, \xi_0)), \qquad \forall\, (\varpi, \xi) \in \mathbb{T}_q \times S_\eta.$$

As a result, $V(\varpi, \xi(\varpi))$ is decreasing, and $V_0 = \lim_{\varpi \to \infty} V(\varpi, \xi)$ exists.

Currently, we assert $V_0 = 0$. If that is not the case, since $V(\varpi, \xi) \ge V_0$, we get

$$\varphi(V(\varpi, \xi)) \ge \varphi(V_0).$$

Consequently, $-\varphi(V(\varpi, \xi)) < -\varphi(V_0)$. As a result, we get

$${}^{C}D^{\alpha}_{\Delta_q,\varpi_0} (V(\varpi, \xi)) \le -\varphi(V_0).$$

By using (20) and (21) with $\upsilon = 0$, we have

$$V(\varpi, \xi(\varpi)) \le V(\varpi_0, \xi_0) - \varphi(V_0) \frac{(\varpi - \varpi_0)^{\alpha}_{q}}{\Gamma_q(\alpha + 1)}.$$

Therefore, because $\lim_{\varpi \to \infty} (\varpi - \varpi_0)^{\alpha}_{q} = \infty$, we get $\lim_{t \to \infty} V(\varpi, \xi) = -\infty$, which is in opposition to the hypothesis that $V(\varpi, \xi)$ is a positive definite.

Therefore, we get

$$\lim_{\varpi \to \infty} V(\varpi, \xi) = V_0 = 0.$$

As a result, $\lim_{\varpi \to \infty} \Xi(\|\xi(\varpi)\|) = 0$, and thus, $\lim_{\varpi \to \infty} \Xi(\|\xi(\varpi)\|) = 0$. □

Lemma 3.1. *If $V(\varpi_0, \xi(\varpi_0)) \geq 0$, then for $0 < \alpha \leq 1$, we have*

$$^{C}D^{\alpha}_{\Delta_q,\varpi_0} V(\varpi, \xi(\varpi)) \leq D^{\alpha}_{\Delta_q,\varpi_0} V(\varpi, \xi(\varpi)), \qquad \forall\, \varpi \geq \varpi_0.$$

Proof. Using Theorem 2.1, we get

$$^{C}D^{\alpha}_{\Delta_q,\varpi_0} V(\varpi, \xi(\varpi)) = D^{\alpha}_{\Delta_q,\varpi_0} V(\varpi, \xi(\varpi)) - \frac{(\varpi - \varpi_0)_q^{-\alpha}}{\Gamma_q(1-\alpha)} V(\varpi_0, \xi(\varpi_0)), \forall\, \varpi \in \mathbb{T}_q, \qquad \varpi \geq \varpi_0.$$

Since $V(\varpi_0, \xi(\varpi_0))$ and $\dfrac{(\varpi - \varpi_0)_q^{-\alpha}}{\Gamma_q(1-\alpha)} \geq 0$, so the result is satisfied. □

Theorem 3.4. *Suppose that,*

1. *If the assumption in Theorem 3.2 is fulfilled by substituting $^{C}D^{\alpha}_{\Delta_q,\varpi_0}$ with $D^{\alpha}_{\Delta_q,\varpi_0}$, then the stationary point of (22) is stable.*
2. *If the assumption in Theorem 3.3 is fulfilled by substituting $^{C}D^{\alpha}_{\Delta_q,\varpi_0}$ with $D^{\alpha}_{\Delta_q,\varpi_0}$, then the stationary point of (22) is uniformly stable.*
3. *If the assumption in Theorem 3.4 is fulfilled by substituting $^{C}D^{\alpha}_{\Delta_q,\varpi_0}$ with $D^{\alpha}_{\Delta_q,\varpi_0}$, then the stationary point of (22) is asymptotically stable.*

Proof. By using Lemma 3.1 and repeating the same justifications as in the proofs of Theorems 3.2 – 3.4, the proof is complete. □

4 Conclusions

The utilization of the Lyapunov sense of stability is observed in diverse fields of scientific investigation and engineering application. The objective of this study is to examine the stability of a dynamic system that incorporates the Caputo delta $q-$fractional derivative. To achieve this, the Lyapunov's direct method is employed. Furthermore, the paper explores the analysis of stability conditions, asymptotic stability, and uniform stability in relation to these systems.

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